IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

On the dimension of a part of the Mandelbrot set

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1989 J. Phys. A: Math. Gen. 22 5249

(http://iopscience.iop.org/0305-4470/22/24/013)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 01/06/2010 at 07:45

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 22 (1989) 5249–5258. Printed in the UK

On the dimension of a part of the Mandelbrot set

Ruud Van Damme

Department of Applied Mathematics, University of Twente, PO Box 217, 7500 AE Enschede, The Netherlands

Received 11 July 1988

Abstract. We investigate one-parameter analytic maps from the complex plane onto itself. We approximate the set of parameter values for the stable periodic orbits, which arise due to subsequent bifurcation from the period-1 orbit, with the aid of normal forms. This approximated set consists of a cactus of touching circles, whose sizes obey a very simple scaling law. From this scaling law the Hausdorff dimension D of the boundary of this approximate set is computed analytically, giving $D = 1.2393\ldots$. Numerical experiments, determining the dimension of the equivalent part of the Mandelbrot set, are consistent with this number. Moreover, this number seems to be independent of the precise form of the map, as predicted by the same analysis.

1. Introduction

Consider an analytic nonlinear map from the complex plane onto itself

$$x' = Cx + f(x) = Cx + a_2x^2 + a_3x^3 + \ldots \equiv H(x) \tag{1.1}$$

where C is a variable complex parameter for which we wish to examine the behaviour of the map, a_i are general complex parameters, which may depend on C. This map has a period-1 fixed point at the origin, which is stable for C values within the circle $|C| = 1$ [1].

From this fixed point periodic orbits bifurcate at C values given by

$$C_0(p/q) = \exp(2\pi i p/q). \tag{1.2}$$

In the following we will write C_0 for the sake of brevity. In (1.2) p and q are two natural numbers which specify the type of orbit: q denotes the period of the orbit; its winding number is p/q. One can assume that $0 < p < q$; furthermore p and q must be relatively prime. A periodic orbit of this kind is called a p/q orbit. The stability of a p/q orbit is determined by the derivative of the map, iterated q times:

$$DH_q = \prod_{k=1}^{q} \left(C + \frac{df}{dx}(x_k) \right) \tag{1.3}$$

where x_k are the p/q-orbit points. This orbit is stable if

$$|DH_q| < 1. \tag{1.4}$$

The problem is to find (approximate) expressions for the regions in C-space, where (1.4) is fulfilled. In section 2 we tackle this problem by normal form techniques. The results in this section show that these stability regions are again circles *and* that the sizes of these regions are independent of the parameters a_k. The set of the stability

0305-4470/89/245249+10$02.50 © 1989 IOP Publishing Ltd

regions corresponding to orbits which emanate due to subsequent bifurcations from the point-1 orbit will be called Mandelbrötchen. By inductive reasoning we arrive at an approximate description of a Mandelbrötchen. In section 3 we compute the Hausdorff dimension of the boundary of this approximate Mandelbrötchen. This will be the main result of this paper. It should be noted that as a Mandelbrötchen is only a part of the full Mandelbrot set, this number is less than the conjectured $D=2$ for the full set. In sections 4 and 5 we describe numerical algorithms and results for the equivalent part of the Mandelbrot set (and related sets). As the results compared with the experiment are beyond expectation, an appendix is devoted to a conjecture on normal forms, which could be the cause of this good agreement.

2. Stability regions

We consider an arbitrary analytic map (1.1) of the complex plane onto itself. It has a fixed point at the origin, with a stability region in the complex C plane given by a circle: $|C|<1$. In the first part of this section we study the stability regions of periodic orbits, directly bifurcating from this period-1 fixed point. In the second part we generalise this result.

For the derivative DH_q of the map H^q (H iterated q times) of a p/q orbit x_k ($k=1\ldots q$), we have

$$DH_q=\prod_{k=1}^{q}\left(C+\frac{\mathrm{d}f}{\mathrm{d}x}(x_k)\right). \tag{1.3}$$

The region of stability is given by

$$|DH_q|<1. \tag{1.4}$$

We want to expand DH_q for C values close to the bifurcation point C_0 (see (1.2)) for given values for p and q. So we let

$$C=C_0+\delta \tag{2.1a}$$

$$|q\delta|\ll 1. \tag{2.1b}$$

Our expansion parameter is given by $|q\delta|$; this choice will be justified at the end of this section. Here we restrict ourselves to the leading order in this parameter; in the appendix we make a comment on the next leading order.

We take the following steps.

(A) First, observe that as a diffeomorphism leaves the eigenvalues of DH_q invariant, we can allow such coordinate transformations. Let

$$x=h(y),\ y=h^{-1}(x) \tag{2.2}$$

then the map (1.1) becomes

$$y'=h^{-1}\circ G\circ h(y). \tag{2.3}$$

As we are interested in the map around its bifurcation point C_0, we can find h such that, according to Poincaré [2], the map is in its normal form: for period p/q we can transform (1.1) into

$$y_{k+1}=Cy_k(1+ay_k^q+\mathrm{O}(y_k^{2q})) \tag{2.4}$$

in which a and b are functions of all the parameters C and a_k, $k=2,3,\ldots$ in (1.1).

(B) Next we define

$$s_m = \sum_{k=1}^{q} y_k^m \qquad m = 1, 2, \ldots \tag{2.5}$$

where y_k are the p/q-orbit points of the map in normal coordinates. Note that the variables s_m $(m = 1, 2, \ldots)$ are not independent; e.g. s_m, with $m > q$, can be expressed algebraically in terms of s_n, with $n \leqslant q$.

Lemma. For any given set $(y_1, \ldots, y_q)$ the averages s_m $(m > q)$ are functions of $s_1, \ldots, s_q$.

$$s_m = w_m(s_1, \ldots, s_q) \qquad \forall m > q \tag{2.6}$$

where w_m is polynomial in each variable $s_1, \ldots, s_q$.

Proof. Observe

$$\prod_{k=1}^{q} (1 - zy_k) = \exp\left(\sum_{k=1}^{q} \log(1 - zy_k)\right)$$
$$= \exp\left(-\sum_{k=1}^{q} \sum_{n=1}^{\infty} z^n y_k^n / n\right) = \exp\left(-\sum_{n=1}^{\infty} z^n s_n / n\right). \tag{2.7}$$

(1) The left-hand side of (2.7) is a polynomial of degree q in z.

(2) The term proportional to z^m of the right-hand side of (2.7) is linear in s_m and contains only s_n with $n \leqslant m$. Combining (1) and (2), one easily proves the lemma by induction. □

(C) Finally we derive a set of equations for the set s_m. By adding both sides of (2.4) for $k = 1 \ldots q$ we get (using $y_{k+1} = y_k$)

$$s_1 = C(s_1 + as_{q+1} + bs_{2q+1} + \ldots). \tag{2.8}$$

By adding both sides of (2.4) squared, for $k = 1 \ldots q$, we get

$$s_2 = C^2(s_2 + 2as_{q+2} + (2b + a^2)s_{2q+2} + \ldots). \tag{2.9}$$

Proceeding in this way by adding both sides of (2.4) raised to the power m, for $k = 1 \ldots q$, we arrive at

$$s_m = C^m s_m + d_{m,q+m} s_{q+m} + d_{m,q+2m} s_{q+2m} + \ldots \qquad \text{for all } m. \tag{2.10}$$

The coefficients $d_{i,j}$ are (in principle) computable coefficients.

As we work here only in first order in δ, one can easily give the solution of the set of equations (2.10). First of all, observe that due to the lemma (see (2.6)), all s_k with $k > q$ vanish in first order in δ. So (2.10) reduces to a set of q equations:

$$s_m = C^m s_m + \text{HOT} = C_0^m s_m + \text{HOT}' \qquad m \leqslant q \tag{2.11}$$

because s_m is at least linear in δ. Equation (2.11) very easily yields within the same approximation order:

$$s_m = 0 \qquad \text{if } m \neq q, \quad s_q \text{ arbitrary}. \tag{2.12}$$

The value of s_q can be determined by observing

$$\frac{y_{k+1}}{y_k} = C(1 + ay_k^q + by_k^{2q} + \ldots) \tag{2.13}$$

and by multiplying all left-hand sides and all right-hand sides of this equation for all orbit points we get, after some algebra,

$$s_q = -\frac{q\delta}{C_0 a} + \mathrm{O}(\delta^2). \tag{2.14}$$

For the derivative we obtain in an analogous way

$$\mathrm{d}H_q = q - C_0^{-1} q^2 \delta + \mathrm{O}(\delta^2). \tag{2.15}$$

So the boundary of the stability region of a period p/q orbit bifurcating from the origin, using (1.4), can be calculated from

$$\mathrm{D}H_q = 1 - C_0^{-1} q^2 \delta = \mathrm{e}^{\mathrm{i}\tau} \qquad \text{with } 0 < \tau \leqslant 2\pi. \tag{2.16}$$

Equation (2.16) represents a circle which touches the original one at an angle $2\pi p/q$ with the real axis and its radius is q^{-2}. Note that (2.1b) is satisfied and that indeed s_k is negligible for $k \neq q$.

In order to generalise this result we make two observations: firstly, the stability region of this newly formed periodic orbit is again a circle; secondly, our result holds for all maps of the form (1.1). Hence it also holds for the same map, iterated q times. Inductively we can repeat the whole argument for any periodic orbit of type $(p/q)^*(p'/q')$ bifurcating from *any* p/q periodic orbit which is a fixed point under the iteration of H q times. For a schematic sketch see figure 1.

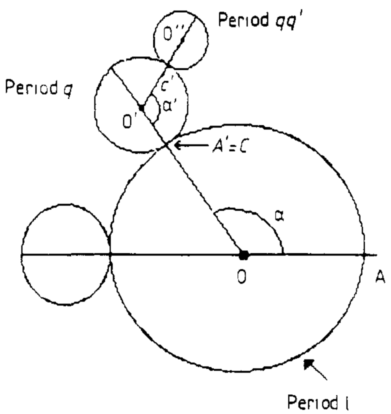

Figure 1. Schematic sketch of the approximation of a Mandelbrötchen. A period p/q bifurcates at the point $C = C_0$ in the complex parameter plane. The angle between OC and OA is $\alpha = 2\pi p/q$. This procedure is repeated with as a new reference the stability region of this new p/q orbit (O$\Rightarrow$O$'$, $C \Rightarrow C'$, $A \Rightarrow A' = C$): a period orbit of type $(p/q)^*(p'/q')$ bifurcates. The new angle between O$'C'$ and O$'A'$ is $\alpha' = 2\pi p'/q'$.

This scaling was observed [3] and explained [4] before; the latter, however, was treated in a completely different way as done here.

In the next section we will determine the dimension of the boundary of this set. In sections 4 and 5 we will compare this with experiments on the genuine Mandelbrot set.

3. The dimension of the boundary of an approximate Mandelbrötchen

In the preceding section we observed that from a periodic orbit of map (1.1) a denumerable set of new periodic orbits bifurcate, which can be labelled by a rational number p/q; the stability region of this new orbit was found to be a circle of radius q^{-2} times the radius of the original circle. As this result (within this approximation) was independent of the map, we could inductively repeat this argument for these new orbits. These facts determine the Hausdorff dimension of the boundary of this approximate set.

We begin by noting that the length of this boundary diverges. Consider all bifurcating orbits from a period-1 orbit. Let its stability region be a circle with radius R. Then there exist exactly $\phi(q)$ birurcating orbits, with a period $q>1$; $\phi(q)$ is the possible number of p values and hence equals the number of integers, which are relatively prime to q. So $\phi(q)$ is the Euler function [5]. The length of this part of the boundary is given by

$$L = 2\pi R \sum_{q=2}^{\infty} q^{-2}\phi(q). \tag{3.1}$$

Since [5]

$$\sum_{q=2}^{\infty} q^{-d}\phi(q) = \frac{\zeta(d-1)}{\zeta(d)} - 1 \tag{3.2}$$

where ζ is the Riemann function, the sum in (3.2) diverges for d equal to 2, and hence L in (3.1) diverges.

For self-similar sets it is relatively easy to determine the Hausdorff dimension. The procedure for computing the dimension is as follows [6].

Introduce a set of similarity transformations v_q $(q = 2, 3, 4\ldots)$ which involve multiplication with r_q (here q^{-2}) and have multiplicity μ_q (here $\phi(q)$). From self-similarity we can deduce, following [6], that the Hausdorff dimension D must satisfy

$$1 = \sum_{q=2}^{\infty} \mu_q r_q^D = \sum_{q=2}^{\infty} q^{-2D}\phi(q) = \frac{\zeta(2D-1)}{\zeta(2D)} - 1. \tag{3.3}$$

Solving this numerically yields for D:

$$D = 1.2393\ldots. \tag{3.4}$$

This is the basic result of this paper. It will be compared with experiments on the dimension of the equivalent part of the Mandelbrot set and related sets.

4. Numerical evaluation of the dimension

The significance of the fact that the dimension of a boundary is larger than 1, is the following. Suppose that we know the shape of the boundary up to a very high precision. If we take a standard step length η $(\eta \ll 1)$ with which we measure the length and plot the logarithm of this length $L(\eta)$ as a function of $\ln(\eta)$, we should find a straight line. The slope of this line is straightforwardly related to the dimension.

Any numerical experiment has the following drawbacks.

(1) The shape of the boundary is only known up to a finite precision. In our experiment we walk along the boundary with a finite step length $\lambda \ll \eta$. As the first

reference point we take the starting point. As soon as the Euclidean distance between the actual point reached and the reference point exceeds η, we add this distance to $L(\eta)$ and take this new point as a new reference.

(2) Due to transient behaviour close to the boundary, escape times can get very long. In fact, by taking a finite number of iterations M, in order to decide whether a point is part of the Mandelbrot set, the set seems to get bigger than it really is.

(3) Everything is being computed with a finite number of digits.

Extending the arguments of [7], we can estimate the errors due to finite λ and transient behaviour. In order to make them of the same order of magnitude, we can argue that M should roughly be λ^{-1}.

The actual effect of transients on the measured dimension compared with the genuine dimension, however, is hard to quantify. Qualitatively we can say that the boundary gets bigger (e.g. one will hit the atennae [1]) but more importantly it will become more vague and hence the measured dimension will be lower than the actual one. At the same time, for very high M the number of digits with which we calculate will influence the results as well. The boundary gets more random and the measured dimension is too high. The net result of these competing errors is not known. In any case, we expect to find for low M a value which is lower than the genuine dimension, since in that case the second error is negligible.

The program was written on a personal computer in order to lower the costs, and we implemented the map in assembler-8087†.

The algorithm used is extremely simple. Suppose we have two points in parameter space $c_{\rm old}$ and $c_{\rm new}$. Assume that

(i) they are at one step (distance λ) from each other;

(ii) they both lie outside the Mandelbrot set;

(iii) they both are within one step from the boundary of the set.

Now take a step with fixed length λ from $c_{\rm new}$, first to the left, then forward, then to the right, and eventually backward, relative to the direction $c_{\rm new} - c_{\rm old}$: the first point (not part of the set) which one encounters, becomes $c_{\rm new}$ (and $c_{\rm new}$ becomes $c_{\rm old}$) (see figure 2). By induction we will find a new pair $(c_{\rm old}, c_{\rm new})$ which satisfies our three assumptions. So if have two proper initial parameter values, satisfying (i), (ii) and (iii), the algorithm finds a path along the boundary.

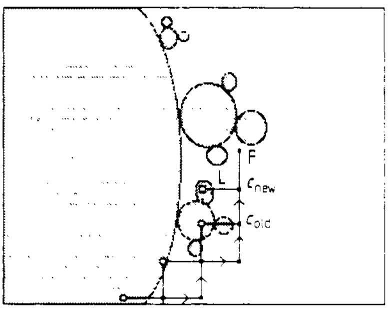

Figure 2. Schematic view of a path along a part of the boundary. The stable set is depicted in grey. White dots indicate misses, black dots are hits. $c_{\rm new}$ and $c_{\rm old}$ of the last step are shown. From $c_{\rm new}$ go to the left (L), which in this case is a miss because the stable region is hit; then proceed in the forward direction (F) and note that now it is a hit because the point is not an element of the stable region. Lines with arrows show the actual path.

† I thank Henk de Leeuw for assembling and inventing this program.

One can convince oneself that this algorithm is 'cusp' proof: it penetrates a cusp but can never get stuck in it (it just turns around); nor can the algorithm get into a loop.

5. Numerical results

We measured the length $L(\eta)$ for length scales η varying from about 10 up to 1000 times λ for some values for M. We used the map

$$x' = A + x^2 \tag{5.1}$$

which can be transformed into (1.1) in a straightforward manner. In our experiments we varied M from 2^{11} up to 2^{16}. In this way we looked at the systematic errors described in the previous section. We took λ fixed: 10^{-5} (in units of A of map (5.1)). We plotted $\ln(L)$ against $\ln(\eta)$ and indeed found approximate straight lines.

Table 1 displays the values for the measured dimension as a function of M.

Observe that as a function of M, the measurements tend to larger predictions for D, which is consistent with our expectation. The best value for D seems to the one at maximal M: $D = 1.239 \pm 0.001$. This is in good agreement with the prediction made in the previous section. We would like to stress, however, that a reliable error estimate is terribly difficult. As already mentioned, the error given above is only the statistical one. Due to non-statistical fluctuations ('lacunarities') we should be more pessimistic on the total error: $D = 1.24 \pm 0.01$ seems to more realistic in view of the results in the table.

Table 1. Measured dimensions ($D \pm$ statistical error) as a function of maximal number of iterations M. D is computed with least square fits.

M	D	Error
2 048	1.221	0.002
4 096	1.220	0.002
8 192	1.223	0.002
16 384	1.233	0.002
32 768	1.240	0.001
65 536	1.239	0.001

A further comment which can be made is the following. In our derivation in § 3, we chose r_q to be q^{-2}, with q the period. Inspection of the Mandelbrot set suggests, however, this factor to be somewhat larger (since $r_2^{-1} \simeq 4.669$, and we also found $r_3^{-1} \simeq 10.03$, $r_5^{-1} \simeq 26$, $r_7^{-1} \simeq 50$; compare [4]). In order to estimate the sensitivity of the dimension on r_q, we compute the dimension when the large-period behaviour of r_q is $(q^2+1)^{-1}$. We then have

$$1 = \sum_{q=2}^{\infty} \mu_q r_q^D = \sum_{q=2}^{\infty} (q^2+1)^{-D} \phi(q). \tag{5.3}$$

Solving this equation numerically yields

$$D = 1.226 \ldots . \tag{5.3}$$

Compared with our numerical result $D = 1.24$, our estimate (3.4) is somewhat better than (5.3), which suggests that for large periods the scaling is indeed q^{-2} rather than $(q^2+1)^{-1}$. Of course a better numerical experiment would be of great value; execution time would be huge, however.

We conclude by noting that (less exact) estimates on the dimension of the boundary for maps with a cubic or quartic term added to the RHS of (5.1), all give consistent values with $D = 1.24 \pm 0.01$.

6. Conclusions

We considered analytic maps from the complex plane onto itself. We investigated the stability regions of periodic orbits which arise due to subsequent bifurcations from some basic periodic orbit (a Mandelbrötchen). The resulting approximate set of this part of the Mandelbrot and related sets consists of circles with well prescribed sizes and positions. The results do not depend on the explicit form of the map. The resulting estimate on the Hausdorff dimension of the boundary of this part of the Mandelbrot set is $D = 1.2393\ldots$, which is in agreement with the best numerical value $D = 1.24 \pm 0.01$. Numerical studies on different maps show that, within the numerical errors, this dimension is independent of the map.

Acknowledgments

The author wishes to thank the Stochting FOM (Fundamental Onderzoek der Materie) for financial support, Henk de Leeuw for indispensable help in computers and Martin Streng for several fruitful discussions. Furthermore the author wishes to express his gratitude to the referees for their valuable remarks.

Appendix

We wish to extend our computation of section 2 to the next leading order. Our starting point is the map in normal form:

$$y_{k+1} = Cy_k(1 + ay_k^q + by_k^{2q} + 0(y_k^{3q})). \tag{2.4}$$

We defined

$$s_m = \sum_{k=1}^{q} y_k^m \tag{2.5}$$

$$C = C_0 + \delta \qquad |q\delta| \ll 1. \tag{2.1}$$

It was shown that, to first order in δ, the variables s_m $(m \neq q)$ vanish. To second order one can easily show that the s_m vanish, unless m equals q or $2q$. The relation between s_q and s_{2q} is derived from (2.7). The term proportional to z^{2q} in the right-hand side of (2.7) should vanish, leading to

$$s_{2q} = s_q^2/q. \tag{A1}$$

We need to know s_q up to order δ^2. Observe that

$$\frac{y_{k+1}}{y_k} = C(1 + ay_k^q + by_k^{2q} + \ldots). \tag{2.13}$$

By multiplying all left-hand sides and all right-hand sides of this equation for all orbit points, we get

$$
\begin{aligned}
1 &= C^q \prod_{k=1}^{q} (1 + a y_k^q + b y_k^{2q}) + \text{HOT} \\
&= C^q\left(1 + a \sum_{k=1}^{q} y_k^q + b \sum_{k=1}^{q} y_k^{2q} + a^2 \sum_{k \neq n} y_k^q y_n^q\right) + \text{HOT} \\
&= C^q(1 + a s_1 + b s_{2q} + a^2(s_q^2 - s_{2q})/2) + \text{HOT}.
\end{aligned} \tag{A2}
$$

Together with (A1) it follows that

$$a s_q = -q\delta C_0^{-1} + q(1 - b/a^2)\delta^2 C_0^{-2} + O(\delta^3). \tag{A3}$$

Similar to (A2) we have

$$DH_q = C^q(1 + a(q+1)s_q + b(2q+1)s_{2q} + a^2(q+1)^2(s_q^2 - s_{2q})/2) + O(\delta^3) \tag{A4}$$

or with (A3)

$$DH_q = 1 - q^2\delta C_0^{-1} + 1/2 q^2 \delta^2 C_0^{-2}(q^2 - q + 2b/a^2) + O(\delta^3). \tag{A5}$$

As the results of the estimate in section 2 on the dimension of a Mandelbrötchen led to a surprisingly good result, we conjecture that the leading-order term in q^4 should vanish for large q: such a term would indeed drastically change this prediction. Hence we are led to

$$\lim_{q \to \infty} q^{-2} b a^{-2} = -\tfrac{1}{2}. \tag{A6}$$

A suggestive plot (and no more than that) is shown in figure 3. We used the simplest mapping in the class of (1.1) (parameter values $a_2 = 1$ and $a_k = 0$ for $k > 2$) and computed the normal forms up to $q = 20$. We plotted the (in general complex) values of $q^{-2}ba^{-2}$ in figure 3. These values tend to a small half-circle around $-\frac{1}{2}$ as q increases. From

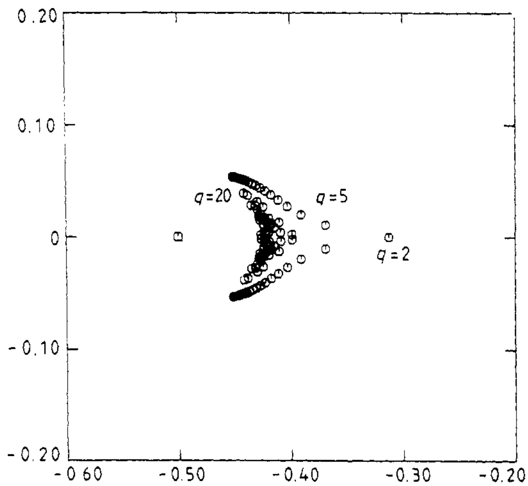

Figure 3. Plot of $q^{-2}ba^{-2}$ for the map $x' = Cx + x^2$, with a and b the first two terms in the normal form expansion (2.4). Note that as q increases its value tend to the point $-\frac{1}{2}$, as conjectured in (A6).

the data it is hard to conclude whether the point $-\frac{1}{2}$ will be reached or not. A final note, supporting our conjecture, is that this figure is rather insensitive for the map chosen: only points for low q values differ recognisably if the map is changed.

References

[1] Peitgen H-O and Richter P H 1986 *The Beauty of Fractals* (Berlin: Springer)
[2] Arnold V I 1988 *Geometrical Methods in the Theory of Ordinary Differential Equations* (Berlin: Springer)
[3] Mandelbrot B B 1982 *The Fractal Geometry of Nature* (San Francisco: Freeman)
[4] Cvitanovic P and Myrheim J 1983 *Phys. Lett.* **94A** 329
[5] Hardy G H and Wright E M 1960 *Introduction to the Theory of Numbers* (Oxford: Oxford University Press)
[6] Falconer K J 1985 *The Geometry of Fractal Sets* (Cambridge: Cambridge University Press)
[7] van Damme R and Valkering T P 1987 *J. Phys. A: Math. Gen.* **20** 4161